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USING PONTRYAGIN MAXIMUM PRINCIPLE FOR PARAMETRICAL IDENTIFICATION OF SHIP MANEUVERING MATHEMATICAL MODEL

Summary. This article proposes usage of Pontryagin maximum principle for parametrical identification of mathematical vessel's model. Proposed method has a special perspective for identification in real time mode, when the parameters identified can be used for forecasting of coming maneuvers.

ИСПОЛЬЗОВАНИЕ ПРИНЦИПА МАКСИМУМА ПОНТРЯГИНА ДЛЯ ПАРАМЕТРИЧЕСКОЙ ИДЕНТИФИКАЦИИ МАТЕМАТИЧЕСКОЙ МОДЕЛИ СУДНА

Аннотация. В статье предложено использование принципа максимума академика Л.С. Понтрягина для параметрической идентификации математической модели судна. Предложенный способ особенно эффективен при идентификации в реальном масштабе времени, когда идентифицированные параметры могут использоваться для прогнозирования ближайших маневров.

There are no disputes about the importance of creation of an adequate mathematical model of a certain vessel. When the model is chosen in some way based on the theory of hydrodynamics, the problem of defining parameters – models coefficients appears. At this stage the priority is given not to theoretical calculation of model's parameters, but to their defining based on vessel field tests. This idea has a special perspective for identification in real time mode, when the parameters identified can be used for forecasting of coming maneuvers.

This process and its result is parametrical identification, as the structure of the model is chosen. In most cases mathematical vessel's model is a system of differential equations of vessel movements, and the parameters are coefficients in the right part of the equations. Usually the coefficients are linear in the right parts, but more complicated cases of parameters inclusion are possible.

The task of parametrical identification is usually formulated as a task of minimisation of some functional in an integral form. If the set of object state variables is given by vector $X = \{x_i\}$, the set of model parameters by vector $\overline{C} = \{C_k\}$, so the condition of functional minimum is as follows:

$$\min\left\{\int_0^{t_f} f(X, dX/dt, d^2X/dt^2, \overline{C}, t)\,dt\right\},\ \overline{C} \in D, \tag{1}$$

Where: D – is some closed area of C model parameters vector variation, and functional under integral in the common case depends both on conditional vector X, and on its first dX/dt and second

d^2X/dt^2 derivatives, here kinematic parameters (V, β, ω) are considered in ship based coordinate system while coordinates of the ship's centre of gravity and her course (X, Y, K) are considered in Earth fixed coordinate system.

Making this function more concrete depends first of all on our ability of monitoring the object, i.e. what state variables we are able to measure. In a perfect case monitoring the vessel's movement we would like to measure 6 variables – three linear accelerations $W = \{w_1, w_2, w_3\}$ and three angular accelerations $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, where coordinates axes $\{X, Y, Z\}$ are chosen for a vessel classically. Monitoring these figures, we could regularly define both kinematic characteristics of six-dimensional movements – linear speed $V = (V_1, V_2, V_3)$ and angular speed $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and linear and angular $\Psi = \{\psi_1, \psi_2, \psi_3\}$ movements, and through them also dynamic characteristics – forces $R = \{R_1, R_2, R_3\}$ and moments $M = \{M_1, M_2, M_3\}$, affecting a vessel. The most important thing is that all these characteristics are defined by integration of accelerations (rather than differentiation of coordinates!). That greatly increases the accuracy of final results.

But such approach is still at the level of ideas, as installation of six-dimensional accelerometers and appropriate processing devices on an ordinary vessel is a rather vague problem. That is why, instead of general problem (1) particular problems of this type are solved, depending on what movement characteristics we can measure directly. For example, if movement speed (lag), coordinates (GPS) and course (gyrocompass) are measured, functional (1) may be written in the following way:

$$\min\left\{\int_0^{t_f}\left[\alpha_1(X-X^\varepsilon)^2+\alpha_2(Y-Y^\varepsilon)+\alpha_3(V-V^\varepsilon)+\alpha_4(K-K^\varepsilon)^2\right]dt\right\}=\min\left\{\int f dt\right\}, \tag{2}$$

where $X^\varepsilon, Y^\varepsilon, V^\varepsilon, K^\varepsilon$ – are figures of these current kinematic characteristics of the ship in the process of movement,

X, Y, V, K – their figures, defined according to the chosen mathematic model and depending on parameters vector $\overline{C}$,

$A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4,\}$ – weight norm $\left(\sum \alpha_k = 1\right)$ vector, the components of which shows us the importance of each one of the kinematic parameter.

But in most cases the problem (2) is taken down to the problem of so-called "differential" identification, i.e. integral minimization, conjugated with the differential equations of vessel's movements. Two-dimensional motion of our ship is described by system of six differential equations:

$$\begin{aligned}
dV/dt &= f_V(V,\beta,\omega,C) & dK/dt &= f_K(V,\beta,\omega,C)\\
d\beta/dt &= f_\beta(V,\beta,\omega,C) & dX/dt &= f_X(V,\beta,\omega,C)\\
d\omega/dt &= f_\omega(V,\beta,\omega,C) & dY/dt &= f_Y(V,\beta,\omega,C)
\end{aligned} \tag{3}$$

In this case we chose the following functional for minimization

$$\min\left\{\int\begin{bmatrix}\alpha_1(dV^\varepsilon/dt - f_V(V^\varepsilon,\beta^\varepsilon,\omega^\varepsilon,C))^2+\alpha_2(d\beta^\varepsilon/dt - f_\beta(V^\varepsilon,\beta^\varepsilon,\omega^\varepsilon,C))^2\\ +\alpha_3(d\omega^\varepsilon/dt - f_\omega(V^\varepsilon,\beta^\varepsilon,\omega^\varepsilon,C))^2\end{bmatrix}dt\right\}=\min\left\{\Phi(V^\varepsilon,\beta^\varepsilon,\omega^\varepsilon,\overline{C})\right\}, \tag{4}$$

where V, β, ω – are important for us parameters.

The problem (4) can be easily presented in a discreet form, replacing the integral by its discreet analogue – by the sum of function under integral in points t_k – time moments of measuring of kinematic parameters of movement $V_k^\varepsilon, \beta_k^\varepsilon, \omega_k^\varepsilon$. After that the task is solved by a traditional OLS (Ordinary Least Squares) method: particular derivatives of the minimized function for the searched parameters are taken as equal to zero. Then we get so called system of normal algebraic equations according to the number of defined parameters:

$$\partial\Phi/\partial Cj = 0,\ j = 1,\ldots 2,\ldots m \tag{5}$$

If the parameters were linear in the model, so the normal system we get is also linear, and from the formal point of view can be easily solved.

But despite this problem being seemingly easy to solve, set like this the implementation of solution results in a whole cluster of problems. Nonobservancy, i.e. impossibility to measure a part of kinematic characteristics, such as β, $d\omega/dt$, dV/dt leads to the necessity to calculate them by differentiation in this or that way. It greatly decreases the accuracy of final results. Besides the matrix of linear problem of type (5) are badly conditioned, and even small deviations of primary data (and in our case they are not small at all!) leads to great deviations of final results in defining Cj parameters. So, these two factors, - low accuracy of primary information and bad matrix conditioning of system (5) make the problem practically incorrect, the result of its solution is dangerous to trust. That is why we would like to get back to problem (2) and look for other approaches for its solution.

Here it is logical to use the Pontryagin maximum principle [1], designed for this kind of tasks. Let's formulate it for our problem. Object's movement is defined by a system of differential equations (3) when choosing coefficients C_j out of a certain closed area D. We should minimize functional (2), where object t_f monitoring time is given. According to the maximum principle this task is equivalent to the following task. Let's consider Hamiltonian (Hamiltonian function) for our problem.

$$H = -f_O + p_v \times f_v + p_\beta \times f_\beta + p_\omega \times f_\omega + p_K \times f_K + p_x \times f_x + p_y \times f_y \ , \tag{6}$$

Here we have new movement characteristics – variables $p_v, p_\beta, p_\omega, p_K, p_x, p_y$ conjugated with certain kinematic parameters of the main equations system, shown by the bottom line indexes. Such variables are described by a system of differential equations, conjugated with the equations of system (3):

$$\frac{dp_v}{dt} = -\partial H / \partial v; \quad \frac{dp_\beta}{dt} = -\partial H / \partial \beta, \quad \frac{dp_\omega}{dt} = -\partial H / \partial \omega,$$
$$\frac{dp_K}{dt} = -\partial H / \partial K; \quad \frac{dp_x}{dt} = -\partial H / \partial X, \quad \frac{dp_y}{dt} = -\partial H / \partial Y; \tag{7}$$

Marginal conditions for main and conjugated variables should be discussed separately in every concrete case. The Pontryagin maximum principle states that the minimum of primary functional is equal to the maximum of Hamiltonian H at any moment t, and its maximal value at the moment $t = t_f$ is equal to zero:

$$H(t_f) = 0 \ , \tag{8}$$

So the task of searching for a minimum is equal to the steering problem: selecting coefficients for C_j model, i.e. steering the model (not the object!) with the help of vector C, we reach the maximum of Hamiltonian H.

It is easy to see, that if the parameters of model are included in it in a linear way, so the Hamiltonian is dependent on them in a linear way. So it's maximum value will actually be the biggest in D-field, as the linear parameter function does not have local maximums inside the field.

The maximum value is reached at the margin of the closed area of possible "controls" for vector C, where hyperplane $H(C) = Const$ either touches the margin of D-field, or some other contact point of hyperplane with a part of the margin.

Particular derivatives $\partial H / \partial C_j$ are not turning into zero, as it would be at the internal maximum point H, they define normal vector cosines direction to hyperplane $H(C) = Const$. Controlling the model, you should move the plane to the normal by increasing H value until it touches the margin of D-field or coincides with a part of it.

What do we usually know about the D-field of permissible values of the model, i.e. the field of permissible model controls from our point of view? Usually hydrodynamic parameters of a model are defined by complicated equations complexes, and even with the similar model structures a greater amount of formulas [3-11] is suggested for C_j assessment. It is the assessment that presumes not precise definition of C_j, but indication of some interval of its possible variation. If the intervals are independent from each other, D-field of permissible values of parameters is a usual hyperparallelepiped in m-dimensional space C.

Our hyperplane $H(C) = Const$ has a general position, i.e. it is not parallel to any of the faces or edges of the parallelepiped. Really, parallelity of a face where $C_j^* = const$ would mean that $\partial H / \partial C_j^* = 0$, and it is possible only if C_j^* is not a part of Hamiltonian *H(C)*. But it is impossible, as any parameter of C_j^* is present in the right part of at least one of differential equations (3). This statement is even more also true for the edges of parallelepiped, where two adjacent faces cross.

It means that, optimal for our controls would be the value of vector C**,** according to the vertex of parallelepiped, where hyperplane touches D-field. For the general plane such vertex always exists and at every moment it is singular, that proves the existence and singularity of solving the set problem. It also proves relay character of the solution as optimal equation – with a certain change of model functioning conditions (sea-going) it is only possible to "jump" in control from one vertex to another.

This conclusion about the relay character of the solution implies another one, which is equally important. It defines the direction, in which theoretical studies in assessment of models parameters should be developed. If in the process of model control there is a "jump" from a certain vertex of D-field to the diagonally opposite vertex, theoretical studies should be directed to narrowing down of D-field in the direction of this diagonal. If such a "jump" is not taking place, only such methods of précising models parameters should be preferred which change D-field in the direction of Hamiltonian *H(**C**)* growth. It immediately allows rejecting the methods which change D-field in the directions, opposite to the above stated.

Let's go over to the concrete equations applied to a vessel's model that realize suggested general positions. Let's choose model structure, that includes traditional linear to drift angle β and angle rate of turn ω parts, and also conjugated non-linearities of type $\beta^2, \omega\beta, \omega\beta^2, \beta^3$. We will also include into identified parameters a coefficient, defining head resistance, proportional to the square of speed and a parameter conjugated with the coefficient of propeller suction. The system of six first degree equations will look the following way:

$$\begin{gathered} dV/dt = C_1V^2\beta\delta - C_2V^2\beta^2 - C_3V\omega\beta - (C_4 + C_6)V^2\beta^2|\beta| + C_5Pe - C_7V^2 = f_v, \\ d\beta/dt = C_1V\delta - C_2V\beta - C_3\omega - C_4V\beta|\beta| = f_\beta, \\ d\omega/dt = C_8V^2\delta + C_9V^2\beta - C_{10}V\omega - C_{11}V\beta^2\omega = f_\omega, \\ dK/dt = \omega = f_K, \\ dX/dt = V\sin(K - \beta) = f_X, \\ dY/dt = V\cos(K - \beta) = f_Y; \end{gathered} \tag{9}$$

The system (9) includes six equations for six vessel state variables $X, Y, V, K, \beta, \omega$ and includes 11 invariables of model parameters $C_j(j = 1\ldots11)$, which should be identified. Let's introduce six conjugated variables $p_v, p_\beta, p_\omega, p_K, p_X, p_Y$ and write down Hamiltonian H for optimality condition in the equation (2):

$$H = -\left[\alpha_1(X - X^\varepsilon)^2 + \alpha_2(Y - Y^\varepsilon)^2 + \alpha_3(V - V^\varepsilon)^2 + \alpha_4(K - K^\varepsilon)^2\right] + p_vf_v + p_\beta f_\beta + p_\omega f_\omega + p_Xf_X + p_Yf_Y, \tag{10}$$

Writing down the expressions in the right part of the system (9) and regrouping the components, let's write (10) in the form of linear function of C_j parameters:

$$H(C)=-\left[\alpha_1(X-X^{\varepsilon})^2+\alpha_2(Y-Y^{\varepsilon})^2+\alpha_3(V-V^{\varepsilon})^2+\alpha_4(K-K^{\varepsilon})^2\right]+p_X V\sin(K-\beta)+$$
$$+p_Y V\cos(K-\beta)+p_K\omega+C_1\left[p_v V^2\beta\delta+p_\beta V\delta\right]+C_2\left[-p_v V^2\beta^2-p_\beta V\beta\right]+C_3\left[-p_v V\beta\omega+p_\beta\omega\right]+$$
$$+C_4\left[-p_v V^2\beta^2|\beta|-p_\beta V\beta|\beta|\right]+C_5\left[p_v P_e\right]-C_6\left[p_v V^2\beta|\beta|\right]-C_7\left[p_v V^2\right]+C_8\left[p_\omega V^2\delta\right]+C_9\left[p_\omega V^2\beta\right]-$$
$$-C_{10}\left[p_\omega V\omega\right]-C_{11}\left[p_\omega V\beta^2\omega\right] \qquad (11)$$

Introduced conjugated variables are described by six differential equations, that come out of correlations (7):

$$dp_v/dt=-\partial H/\partial V=-\left\{\begin{array}{l}-2\alpha_3(V-V_\varepsilon)+p_X\sin(K-\beta)+p_Y\cos(K-\beta)+C_1\left[p_v 2V\beta\delta+p_\beta\delta\right]+\\+C_2\left[-p_v 2V\beta^2-p_\beta\beta\right]+C_3\left[-p_v\beta\omega\right]+C_4\left[-p_v 2V\beta^2|\beta|-p_\beta\beta|\beta|\right]-\\-C_6\left[p_v 2V\beta^2|\beta|\right]-C_7\left[p_v 2V\right]+C_8\left[p_\omega 2V\delta\right]+C_9\left[p_\omega 2V\beta\right]-C_{10}\left[p_\omega\omega\right]-\\-C_{11}\left[p_\omega\beta^2\omega\right]\end{array}\right\},$$

$$dp_\beta/dt=-\partial H/\partial\beta=\left\{\begin{array}{l}-p_X V\cos(K-\beta)+p_Y V\sin(K-\beta)+C_1\left[p_v V^2\delta\right]+\\+C_2\left[-p_v 2V\beta^2-p_\beta V\right]+C_3\left[-p_v V\omega\right]+\\+C_4\left[-p_v V^2 3\beta^2 sign(\beta)-p_\beta 2V|\beta| sign(\beta)\right]-\\-C_6\left[p_v V^2 3\beta^2 sign(\beta)\right]+C_9\left[p_\omega V^2\right]-C_{11}\left[p_\omega 2V\beta\right]\end{array}\right\}$$

$$dp_\omega/dt=-\partial H/\partial\omega=p_K+C_3\left[-p_v V\beta-p_\beta\right]-C_{10}\left[p_\omega V\right],$$

$$dp_K/dt=-\partial H/\partial K=2\alpha_4(K-K_\varepsilon)^2+p_X V\cos(K-\beta)-p_Y V\sin(K-\beta),$$

$$dp_X/dt=-\partial H/\partial X=2\alpha_1(X-X^{\varepsilon}),$$

$$dp_Y/dt=-\partial H/\partial Y=2\alpha_2(Y-Y^{\varepsilon}). \qquad (12)$$

Solution of first degree systems (9) and (12) gives 12 arbitrary invariables, for definition of which 12 marginal conditions are necessary. Five of them for initial time t_0 (without loosing generality it can be accepted as zero) are obtained by equating parameters of model state to the value, obtained in monitoring the object:

$$X=X^{\varepsilon},\ Y=Y^{\varepsilon},\ V=V^{\varepsilon},\ K=K^{\varepsilon},\ \beta=\beta^{\varepsilon},\ \omega=\omega^{\varepsilon}. \qquad (13)$$

At the moment $t = t_f$ summarizing object's movement monitoring final marginal conditions are much more complicated. Here one can require only a certain approximation degree of some parameters of model and object state, for example:

$$\begin{array}{c}(X-X^{\varepsilon})^2+(Y-Y^{\varepsilon})^2\le r^2,\\(V-V^{\varepsilon})^2\le\varepsilon_V^2,\\(K-K^{\varepsilon})^2\le\varepsilon_K^2,\\(\omega-\omega^{\varepsilon})^2\le\varepsilon_\omega^2,\\t=t_f;\end{array} \qquad (14)$$

where $\varepsilon_V,\varepsilon_K,\varepsilon_\omega$ - are set up by us deviation values of final motion parameters.

These conditions for getting into a certain field of parameters can be reduced to setting the border of the field, changing in (14) in equations for equations. This comes out of the idea, which coming into the field with constant change of state is possible only when crossing its margins. Let's name marginal conditions (14) with a change of inequalities for corresponding equations. These equations imply so called transversality conditions, including conjugated variables. In the general form they look like this:

$$P_i+\sum\Lambda_j(\partial G_j/\partial X_i)=0; \qquad (15)$$

Here $G = G(X_1, X_2, \dots X_n) = 0$ – is the marginal condition, X_i – parameters of the object state, p_i – corresponding conjugated variables, and Λ_j – new unknown invariables. Considering marginal conditions (14a), we get the following view of transversality conditions in accepted indexes not by the number, but by the name of state variable:

$$\begin{gathered} P_X + \Lambda_1 2(X - X^{\varepsilon}) = 0, \\ P_v + \Lambda_1 2(V - V^{\varepsilon}) = 0, \\ P_Y + \Lambda_1 2(Y - Y^{\varepsilon}) = 0, \\ P_{\omega} + \Lambda_1 2(\omega - \omega^{\varepsilon}) = 0, \\ P_K + \Lambda_1 2(K - K^{\varepsilon}) = 0, \\ P_{\beta} = 0,\ t = t_f; \end{gathered} \tag{16}$$

Appearance of four new invariables will lead to the common number of unknown invariables, that is equal to 16. So long we only have 15 conditions: 5 marginal conditions (13), four marginal conditions (14) and six transversality marginal conditions (16). One missing marginal condition we will get from the consideration that Hamiltonian maximum (11) for the final moment of time turns into zero according to the maximum principle:

$$H(t = t_f) = 0; \tag{17}$$

This equation includes both variables of condition and conjugated variables. The problem is closed by this marginal condition and we can try to make attempts to find solution. Existence and singularity of this solution in our case has been shown earlier as existence of the single contact point of the general position hyperplane.

$H = const$ with the control field D in one of its vertices. As for finding the solution, it can be obtained only by numerical methods [2], considering the general complexity of this problem. Such complexity is first of all connected with the fact that the problem is a two point Cauchy problem: a part of marginal conditions is set for $t = 0$, and the other part for $t = t_f$, that requires testing in obtaining solution. But we undertake attempts to find such solution.

In conclusion let's consider example of using Pontryagin maximum principle method for the simple acceleration(stopping) of the ship on a straight-line trajectory motion task. In this case equations system (9) solved by two differential equation:

$$\begin{gathered} \frac{dV}{dt} = C_0 P_e - C_1 V^2, \\ \frac{dx}{dt} = V; \end{gathered} \tag{18}$$

Here mathematical ship's model contains only two constants C_0 and C_1. Let's choose for minimization of functional acceleration time or stopping time i.e. let's consider classic speed-in-action problem

$$\min \left\{ \int_0^{t_f} dt \right\} = \min \{ t_f \}; \tag{19}$$

In this case Hamiltonian will be as follows:

$$H = -P_0 + P_V (C_0 P_e - C_1 V^2) + P_X(V); \tag{20}$$

and equation for conjugated variables takes the form:

$$\begin{gathered} \frac{dP_V}{dt} = -\frac{\partial H}{\partial V} = P_V C_1 2V - P_x, \\ \frac{dP_X}{dt} = -\frac{\partial H}{\partial X} = 0, \text{ т.е. } P_X = P_V = const; \end{gathered} \tag{21}$$

Due to similarity of input P_0, P_V, P_X in H we can consider $P_0=1$.

Partial derivatives of Hamiltonian H with respect to ship's model are:

$$\frac{\partial H}{\partial C_0} = P_V P_e, \tag{22}$$

$$\frac{dH}{dC_1} = -2P_V V; \tag{23}$$

Let's carry out analysis for acceleration:

$$P_e > 0,$$

$$V > 0,$$

$$sign\left[\frac{\partial H}{\partial C_0}\right] = sign\left[P_V\right], \tag{24}$$

$$sign\left[\frac{\partial H}{\partial C_1}\right] = -sign\left[P_V\right]; \tag{25}$$

These marks are defining position of hyperplane $H=const$ fractional to closed area D and defining the normal line $\overline{n}$ to this hyperplane in growing direction of H. Here, in our case of two-parameters ship's model, these properties easily demonstrated on fig. 1.

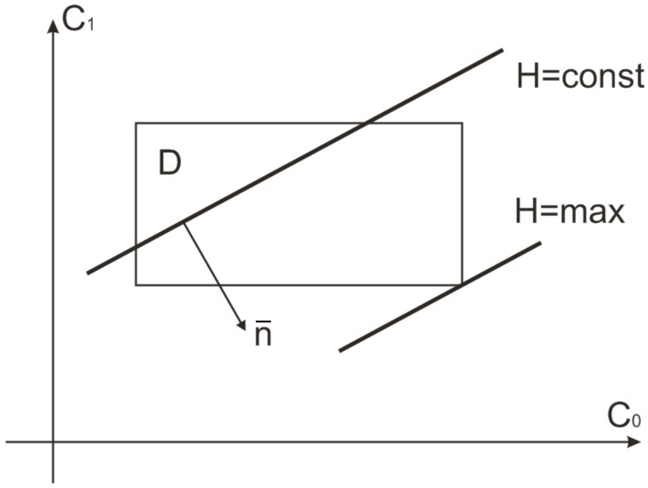

Fig. 1. Position of hyperplane *H*

Рис. 1. Расположение гиперплоскости *H*

Function H reach her maximum in the down right corner of D-area. Therefore during calculation of parameters of this ship's model, among all existing alternatives, we are to choose those which makes C_0 maximum and C_1 minimum. Thus, the proposed development may be useful in the design of navigational instruments, such as [14].

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