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MINIMIZATION OF BUS STOP NUMBER ON A BUS STATION

Summary. A bus station contains several bus stops. Only one bus can occupy a single bus stop at a time. Buses of many trips arrive to the bus station during the day (or during another considered period) and every bus occupies a bus stop for a certain time interval. The set of available bus stops is limited. This paper studies a problem how to assign a bus stop to every bus trip in order to minimize the number of assigned bus stops and in order to comply several additional conditions. Several approaches to this problem are presented. These approaches differ according to considered additional conditions.

MINIMALIZACJA LICZBY PLATFORM NA STACJI AUTOBUSOWEJ

Streszczenie. Na stacji autobusowej może znajdować się kilka platform. W tym samym czasie przy jednej platformie może znajdować się tylko jeden autobus. W ciągu dnia na stację autobusową przyjeżdżają autobusy z różnych połączeń i każdy z nich zajmuje platformę przez określony czas. Ten artykuł ma na celu pokazanie problemu przyporządkowania platform do wszystkich połączeń i jednoczesnej minimalizacji liczby platform przy spełnieniu określonych warunków. Prezentowane są różne sposoby rozwiązania problemu. Każdy ze sposobów różni się w zależności od dalszych warunków.

1. INTRODUCTION

A bus trip (or a bus journey) is a one-way movement of a bus along a route between two terminal points. Bus trips are arranged into lines. A line is a set of trips having the same or similar route. As a rule, a line contains trips having two opposite directions – back and forth. A subset of trips of a line having the same direction is called a line direction.

Buses of several trips from various lines and many directions arrive and depart from a bus station in departure times determined by the time table. A bus station contains several bus stops. Each bus stop has capacity equal to 1, i.e. only one bus can stay at one bus stop at a time. In the ideal case, one bus stop is assigned only to the trips of one particular direction of one line. This requirement is motivated by the effort to make the orientation at the bus station for passengers as easy as possible. But the number of bus stops at the bus station is limited and is less than the number of all line directions of all bus lines. Therefore more line directions have to share the same bus stop.

In a railway station the number of platforms is very small (in comparison to a bus station) and therefore the problem of train trips – platform assignment takes neither directions nor train lines into consideration. Therefore the trip-platform assignment problem for a railway station is simpler from the mathematical point of view.

2. CHAPTER UNCODITIONAL BUS STOP MINIMIZATION

Let us study the case in which there are no restrictions to bus stop assignment to trips. This situation can occur on small bus stations with a small number of bus stops or on railway stations. Let $T = \{t_1, t_2, \cdots, t_n\}$ be the set of all trips with corresponding arrival times $a_1, a_2, \cdots, a_n$ and departure times $d_1, d_2, \cdots, d_n$. Trip t_i occupies the bus stop in the interval (a_i, d_i).
Define

$$c_{ij} = \begin{cases} 0 & if \quad d_i \le a_j \\ 1 & otherwise \end{cases} \tag{1}$$

Let x_{ij} be a binary decision variable saying that trips t_i, t_j share the same bus stop immediately t_j after t_i if and only if $x_{ij} = 1$ and $c_{ij} = 0$. Let the set of values $\{x_{ij}\}_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,n}}$ define for every t_i its successor t_j. The trip t_j is a successor of t_i iff $x_{ij} = 1$. If in this situation $c_{ij} = 0$, the trips t_i, t_j can share the same bus stop, but if $c_{ij} = 1$, the trip t_j needs another bus stop. So the required number of bus stops can be evaluated as $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}.x_{ij}$. Thus the problem of unconditional bus stop minimization can be formulated as follows:

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}.x_{ij} \tag{2}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1,2,\ldots,n \tag{3}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1,2,\ldots,n \tag{4}$$

$$x_{ij} \in \{0,1\} \tag{5}$$

We see that we got a classical assignment problem, which can be solved, in polynomial time.

If the optimum solution of problem (2) – (5) gives less or equal platforms than available number, we can try to find such an assignment, which allows for train or bus delay δ.
Define

$$c_{ij}^{*} = \begin{cases} 0 & if \quad d_i \le a_j - \delta \\ 1 & otherwise \end{cases} \tag{6}$$

and solve the following problem: Minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^{*}.x_{ij}$ subject to (2) – (5).
If the resulting number of platforms is greater than available number, try this procedure with smaller δ.

However, this optimization does not take line directions into account at all. But it gives the exact lower bound for number of bus stops and the optimum solution can be used as a starting solution for further heuristic algorithms, which can improve some objective function expressing additional special requirements for bus stop assignment.

There exists a simpler way how to compute the lower bound for the number of used bus stops.

For every trip t_i we can construct the function $f_i(x)$ (called deficit function) with the domain $\langle 0,1440 \rangle$ (the set of minutes of the day) defined

$$f_i(x) = \begin{cases} 1 & \text{if a bus of trip } t_i \text{ needs a bus stop in time } x \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

It holds $f_i = 1$ if and only if $x \in \langle a_i, d_i \rangle$ where a_i is arrival time and d_i is departure time of trip t_i.

If $D$is a line direction (i.e. a set of trips of the same line and the same direction) the deficit function of line direction D is defined as

$$f_D(x) = \sum_{i \in D} f_i(x) \quad (8)$$

Let $f_1, f_2, \cdots, f_n$ are deficit functions for trips $t_1, t_2, \cdots, t_n$. Denote by $F(x)$the sum of all deficit functions, i.e.:

$$F(x) = \sum_{i=1}^{n} f_i(x) \ . \quad (9)$$

$F(x)$ is called cumulative deficit function. The lower bound for the number of bus stops can be computed as $\max_{x=0,1,\dots,1440}\{F(x)\}$. However, this simple approach does not give us any bus stop assignment.

In many cases arrival times are not given. In these cases we can assume that $a_i = d_i - \delta$, where δ is the time needed for boarding the bus (in most cases $\delta \in \langle 5,10 \rangle$). An example of a real cumulative deficit function for bus station Zvolen and for $\delta = 10$ is in Fig.1.

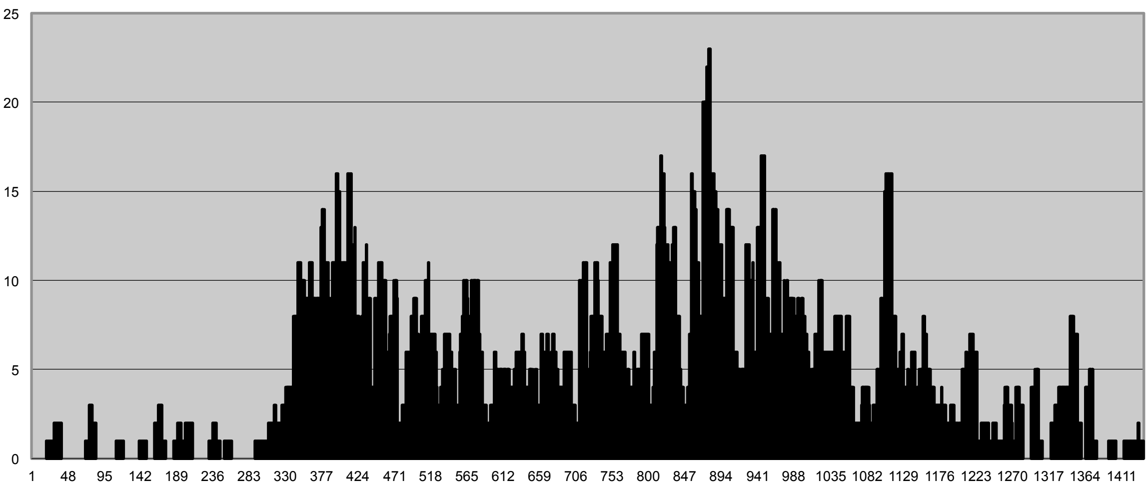

Fig. 1. Cumulative deficit function for bus station Zvolen
Rys. 1. Łączny deficyt funkcji dla stacji autobusowej Zvolen

3. CHAPTER GRAPH COLORING MODEL FOR THE BUS STATION PROBLEM

In this section we will minimize the number of used bus stops with the only requirement that all trips of the same line directions are assigned the same bus stop.

Let p, q be two line directions. It is easily seen that all trips of two line directions p, q can share the same bus stop if and only if $f_p(x) + f_q(x) \leq 1$ for all $x \in \langle 0,1440 \rangle$ – in this case we will say that the line directions p, q are compatible. More general, the set S of line directions can share the same bus stop if and only if the sum of their deficit functions is less or equal to 1, i.e. if $\sum_{s \in S} f_s(x) \le 1$ for all $x \in \langle 0{,}1440 \rangle$ – in this case we will say that S is a set of compatible line directions.

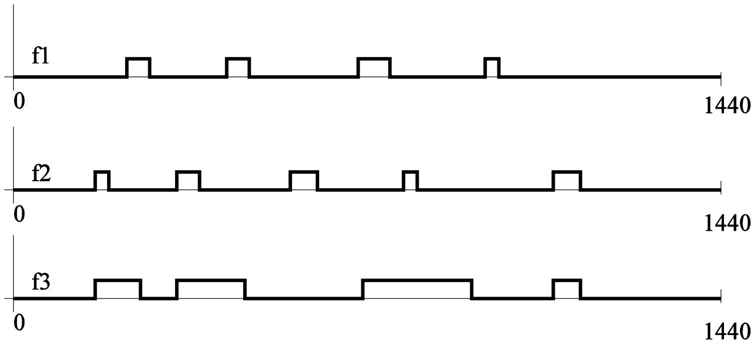

Fig. 2. Deficit functions $f_1(x)$, $f_2\ (x)$ compatible, $f_3(x)$ incompatible with both $f_1(x)$, $f_2(x)$
Rys. 2. Niedobór funkcji $f_1(x)$, $f_2\ (x)$ zgodny, $f_3(x)$ niezgodny z $f_1(x)$, $f_2(x)$

Let S be a set of all line directions, which use the bus station currently under consideration. Define a graph$G = (V, E)$ with the vertex set $V = S$ and with the edge set E containing all unordered pairs (p, q) of uncompatible line directions. We can formulate the problem of minimization of used bus stops as the following graph coloring problem:

To assign a color (= bus stop) to every vertex (= line direction) of graph $G = (V, E)$ such that no adjacent vertices (= no incompatible line directions) are assigned the same color (= the same bus stop) and such that the number of used colors (= the number of used bus stops) is minimal.

The assignment of colors to vertices of the graph G such that no adjacent vertices are assigned the same color is called the proper graph coloring. The chromatic number of a graph, denoted $\chi(G)$, is the minimum number of different colors required for a proper graph coloring. The graph coloring problem is to find a proper graph coloring with $\chi(G)$ colors.

The following exact graph coloring algorithm comes from the Demel's book 1. Let V_x be the set of all neighbours of the vertex x in graph G. Assume that $V_x = \{1,2, \dots, n\}$. Let $B[j]$ be an actual color of vertex j.

Set
$$P(x) = V_x \cap \{1,2, \dots, x-1\}$$
$$F(x) = \min \ \{i \mid \ 1 \le i, \quad \forall j \in P(x)\ i \ne B[j]\}$$
$$G(x) = \min \ \{i \mid \ B[x] < i, \forall j \in P(x)\ i \ne B[j]\}$$

$F(x)$ is the lowest feasible color number for vertex x. Provided that the vertex x is colored with color $[x]$, $G(x)$ is the lowest feasible color number greater then B[x] which can be used for vertex x.

- **Step 0.** Set $B[1] = 1$ and sequentially for every $x = 2;\ 3, \dots, n$ set $[x] = F(x)$.
- **Step 1.** Set $FMAX = \max_{1 \le x \le n}\{B[x]\}$. Copy array $B[\,]$ into array $RECORD[\,]$.
- **Step 2.** Find in array $B[\,]$ the least y such that $B[y] = FMAX$.
- **Step 3.** Set $x = \max_{k \in P(y)}\{k\}$
- **Step 4.** If x = 1, STOP. Chromatic number of graph G is = FMAX
  and the corresponding graph coloring of G is in array RECORD[ ].
- **Step 5.** If $G(x) \ge FMAX$ or $if\ G(x) > (\max_{1 \le i < x}\{B[i]\} + 1)$,
  set $x := x - 1$ and Goto Step 4.
  Otherwise set $B[x] = G(x),\ z = x + 1$.
- **Step 6.** Set $[z] = F(z)$. If $B[z] \ge FMAX$, set $y = z$ and Goto Step 3.
  If $z < n$, set $z := z + 1$ and repeat Step 6.
  If $z = n$, we have a new better solution. Goto Step 1.

Another good graph coloring algorithm was designed by M. A. Trick – see [3]. This algorithm makes use of independent sets and seems to be faster than that of Demel. Nevertheless, graph coloring problem is NP – hard. One good property of Demel's algorithm is that after Step 1 array $RECORD[\,]$ contains a suboptimal solution.

4. CHAPTER ANOTHER APPROACH TO BUS STOP ASSIGNMENT

Practical experiences show that personal bus transport providers do not strictly follow the rule that all trips of one line direction have to share the same bus stop. This relaxation can lead to even better solution. However, we still have to endeavour to minimize number of cases when a bus arrives to bus stop with different line direction.

Starting solution for this approach can be obtained as a solution of the assignment problem (2) – (5). As a result we will obtain the minimum number of bus stops k and for every bus stop i the corresponding set of trips $T_i = \{ t_{i1}, t_{i2}, \cdots, t_{im_i} \}$ which will share the bus stop i.

Let p, q be two trips sharing the same bus stop. Let us define the inconvenience of trips p, q as

$$dif(p,q) = \begin{cases} 0 & if\ \ p, q \quad have\ the\ same\ line\ direction \\ 1 & otherwise \end{cases}$$

Let us define the inconvenience of the set T_i as

$$c(T_i) = \sum_{p \in T_i} \sum_{q \in T_i} dif(p,q)$$

and the total inconvenience of a bus station as

$$C(T_1, T_2, \cdots, T_k) = \sum_{i=1}^{k} c(T_i). \tag{10}$$

The minimization of $C(T_1, T_2, \cdots, T_k)$ tends to assign all trips having the same line direction to the same bus stop since it makes the number of interactions of trips with different line directions in the same bus stop as low as possible. If the bus station has more bus stops than the minimum number k obtained from solution of (2) – (5), we can make another starting solution from solution $T_1, T_2, \cdots, T_k$ by decomposing some sets into two or more pieces. We can use also more complicated objective function for $c(T_i)$ which can express incompatibility of trips in a quantitative way.

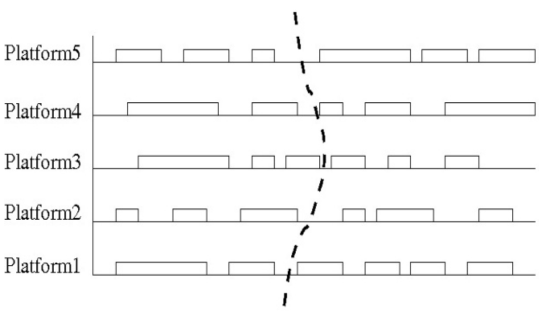

Fig. 3. Decomposition of sets of trips on platforms into heads and tails
Rys. 3. Rozłożenie zestawów podróży na peronach na przody i tyły

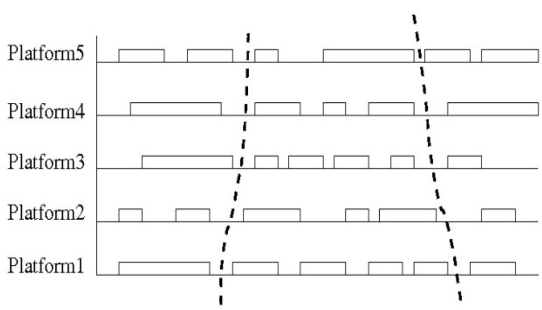

Fig. 4. Decomposition of sets of trips on platforms into heads, mids and tails
Rys. 4. Rozłożenie zestawów podróży na peronach na Przody, środki i tyły

The following algorithm showed to be extraordinary good for minimization of objective function $C(T_1, T_2, \ldots, T_k)$ defined in (10).

- **Step 0.** Solve assignment problem (2) – (5). The result is k - the minimum number of bus stops k and for every bus stop i a set of trips $T_i = \{ t_{i1}, t_{i2}, \cdots, t_{im_i} \}$ assigned to bus stop.

- **Step 1.** For $r = 0$ to 1400 do:

A) Decompose every set T_i into two sets: $HEAD(T_i)$ – the subset of trips departing in time r or sooner and $TAIL(T_i)$ – the subset of trips departing later than r.

B) For every pair (i, j) do:

- Compute C_{ij} $-$ the inconvenience of the set $HEAD(T_i) \cup TAIL(T_j)$.
  If $HEAD(T_i) \cup TAIL(T_j)$ contains incompatible trips then set $C_{ij} = \infty$.
- Find an optimum assignment of heads to tails with respect to cost C_{ij}.
- Create new sets T by combining heads and tails according to resulting assignment.

- **Step 2**. For $r = 0$ to 1399 do: For $s = r + 1$ to 1440 do:

A) Decompose every set T_i into three sets: $HEAD(T_i)$ – the subset of trips departing sooner than r, $TAIL(T_i)$ – the subset of trips departing later than r and $MID(T_i)$ – the subset of trips departing in closed interval $\langle r, s\rangle$.

B) For every pair (i,j) do:

- Compute C_{ij} - the inconvenience of the set $HEAD(T_i) \cup MID(T_j) \cup TAIL(T_i)$.
  $HEAD(T_i) \cup MID(T_j) \cup TAIL(T_j)$ contains incompatible trips then set $C_{ij} = \infty$.
- Find an optimum assignment of heads to tails with respect to cost C_{ij}.
- Create new sets T by combining heads, mids and tails according to resulting assignment.

- **Step 3**. If no improvement of total inconvenience occurred neither in Step 1. nor in Step 2. – STOP. Otherwise GOTO Step 1.

5. CONCLUSION

The aforementioned techniques were applied to bus station of the Slovak town Zvolen. This bus station consisted of 26 bus stops. The question was whether this number could be decreased to 20 in order to use 6 bus stops for other purposes. The trips were given by their departure times d_i, corresponding arrival times were calculated as $a_i = d_i - \delta$ for $\delta = 5$. Graph coloring approach did not offer satisfactory solution. Unconditional bus stop minimization resulted in 13 bus stops. Afterwards the procedure described in chapter 4. was applied. It gave an outstanding result for 20 bus stops where trips of one line direction were assigned to the same bus stop with very rare exceptions.

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